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Interactions among periodic waves and solitary waves for a higher dimensional system

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Abstract

The general variable separated approach is successfully extended to the (2+1)-dimensional physical model and we obtain a universal formula involving arbitrary number of variable separated functions. Because of the existence of the arbitrary functions in the universal formula, many new types of structures of periodic waves, for example, the periodic–periodic interaction waves, periodic–kink interaction waves, periodic–peakon interaction waves, periodic–compacton interaction waves, periodic–foldon interaction waves, etc, are investigated both analytically and graphically. Some novel features or interesting behaviours are revealed.

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1. Introduction

In the study of nonlinear science, soliton theory plays a very important role and has been applied in almost all the natural sciences especially in all the physics branches such as condensed matter physics, field theory, fluid dynamics, plasma physics and optics, etc [1]. It is known that for the (2+1)-dimensional integrable models, there are many more abundant localized structures than in (1+1)-dimensional case because some types of arbitrary functions can be included in the explicit solution expression [2–4]. In the traditional treatment of a nonlinear system, one usually studies the interaction behaviours among solitons (or solitary waves) in respect that many methods can provide exact explicit multiple soliton (or solitary wave) solutions. However, there are few works in the literature to study the interactions among (elliptic) periodic waves and/or between the periodic and solitary waves because of the difficulties to find exact and explicit multiple (elliptic) periodic wave solutions and/or periodic-solitary wave solutions though one knows a single solitary wave solution can be considered as a limit case of a single periodic wave solution.

Motivated by these reasons, we take the (2+1)-dimensional Burgers equation

$$u_t - uu_y - avu_x - bu_{yy} - abu_{xx} = 0, \tag{1a}$$

$$u_x = v_y, \tag{1b}$$

where a and b are the arbitrary constants, as a concrete example to studying the interactions among periodic waves and solitary waves. An equivalent form of the potential Burgers equation (1) is derived from the generalized Painlevé integrable classification [5]. In [6], one of the present authors (C-L Bai) has studied the initial value problem and some exact solutions of equation (1).

In section 2, we apply a general variable separated approach to solve the (2+1)-dimensional Burgers equation and obtain its exact and explicit general solution. In section 3, some concrete exact solutions such as the periodic–periodic, periodic–kink, periodic–peakon, periodic–compacton and periodic–foldon interaction are graphically displayed. A brief discussion and summary are given in the final section.

2. Variable separated solutions for the (2+1)-dimensional Burgers equation

There was a wealth of approaches for finding special solutions of the nonlinear partial differential equation (PDE), such as the inverse scattering method, Darboux transformation, the hyperbolic tangent method, the generalized variable-coefficient algebraic method, etc. All these methods are described in [1–4, 7–10]. Recently, Lou and Tang [2, 3] have proposed a multilinear variable separation approach (MLVSA) to search for the exact solutions of the higher dimensional, specially (2+1)-dimensional, nonlinear PDEs. More recently, Tang and Lou have proposed a more general variable separation approach (GVSA) for several (2+1)-dimensional integrable models including the DLWE, the BKK system [11]. In this paper, we develop and apply the GVSA to the (2+1)-dimensional Burgers equation and study interactions among periodic waves and solitary waves.

According to the standard truncated Painlevé expansion, there is the Bäcklund transformation

$$u(x, y, t) = \frac{2bf_y}{f} + u_0(x, y, t), \tag{2a}$$

$$v(x, y, t) = \frac{2bf_x}{f} + v_0(x, y, t), \tag{2b}$$

for the (2+1)-dimensional Burgers equation, where $\{u_0, v_0\}$ is an arbitrary known seed solution of equation (1). Substituting equations (2) into equations (1), two equations reduce to a single equation

$$\begin{aligned} u_0\left(ff_{yy} - f_y^2\right) + av_0(ff_{xy} - f_x f_y) + u_{0y} ff_y + au_{0x} ff_x \\ - (f\partial_y - f_y)(f_t - bf_{yy} - abf_{xx}) = 0. \end{aligned} \tag{3}$$

Evidently, equation (1) possesses a trivial seed solution:

$$u_0 = 0, \qquad v_0 = v_0(x, t), \tag{4}$$

where v_0 is the arbitrary function of the indicated variable. Because of equation (4), equation (3) becomes a linear equation

$$f_t - bf_{yy} - abf_{xx} - av_0 f_x - \beta f = 0, \tag{5}$$

with $\beta \equiv \beta(x,t)$ being an integration function. In order to find more exact solutions of the (2+1)-dimensional Burgers equation, we use the GVSA to equation (5) and take the general ansätz for the function f

$$f = q_0 + \sum_{i=1}^{N} p_i q_i, \tag{6}$$

where q_i, $i = 0, 1, \ldots, N$ and p_i, $i = 1, 2, \ldots, N$ are functions of $\{y, t\}$ and $\{x, t\}$, respectively.

Substituting ansätz (6) into (5) and with the help of the computer algebras (say, MAPLE), we can get the following relations of q_i and p_i:

$$p_{it} = \left(ab\partial_x^2 + av_0\partial_x + \beta - C_i\right)p_i, \qquad i = 1, 2, \ldots, N, \tag{7}$$

$$q_{it} = bq_{iyy} + C_i q_i, \qquad i = 0, 1, 2, \ldots, N, \tag{8}$$

where (C_i, $i = 0, 1, 2, \ldots, N$) being arbitrary functions of t. The corresponding solution for the quantity V ($V \equiv u_{xy}/b \equiv v_{yx}/b$) can be written as

$$V = \frac{-2\sum_{i=1}^{M} p_{ix}q_{iy}}{q_0 + \sum_{i=1}^{M} p_i q_i} + \frac{2\sum_{i=1}^{M} p_{ix}q_i\left(q_{0y} + \sum_{j=1}^{N} p_j q_{jy}\right)}{\left(q_0 + \sum_{i=1}^{M} p_i q_i\right)^2}. \tag{9}$$

For simplicity to study the interaction properties among periodic waves and solitary waves for the potential V of the (2+1)-dimensional Burgers equation, we fix

$$M = N = 1, \qquad p_1 = p,$$

in relation (9); then formula (9) becomes

$$V = \frac{2p_x(q_1 q_{0y} - q_0 q_{1y})}{(q_0 + pq_1)^2}, \tag{10}$$

where q_0 and q_1 are the arbitrary functions of $\{y, t\}$ and p is an arbitrary function of $\{x, t\}$.

3. Interactions among periodic waves and solitary waves for the (2+1)-dimensional system

3.1. Interactions among doubly periodic waves

It is known that for the nonlinear systems, the doubly periodic wave structures can usually be expressed by means of the Jacobi elliptic functions with constant module. For instance, if we take

$$p = \mathrm{sn}(\xi_1, m_1) + \mathrm{sn}(\xi_2, m_2), \qquad \xi_1 = k_1 x + \omega_1 t, \qquad \xi_2 = k_2 x + \omega_2 t,$$

$$q_0 = \alpha + \mathrm{sn}(\eta_0, n_0), \qquad q_1 = \mathrm{sn}(\eta_1, n_1), \qquad \eta_0 = l_0 y + c_0 t, \qquad \eta_1 = l_1 y + c_1 t,$$

where sn is the Jacobi elliptic sine function, k_1, ω_1, k_2, ω_2, l_0, c_0, l_1, c_1, m_1, m_2, n_0 and n_1 are arbitrary constants, constant α guarantees that the solution has no singularity, then we have

$$V = 2(k_1\,\mathrm{cn}(\xi_1, m_1)\mathrm{dn}(\xi_1, m_1) + k_2\,\mathrm{cn}(\xi_2, m_2)\mathrm{dn}(\xi_2, m_2)) \times \frac{(l_0\,\mathrm{sn}(\eta_1, n_1)\mathrm{cn}(\eta_0, n_0)\mathrm{dn}(\eta_0, n_0) - l_1(\alpha + \mathrm{sn}(\eta_0, n_0))\mathrm{cn}(\eta_1, n_1)\mathrm{dn}(\eta_1, n_1))}{(\alpha + \mathrm{sn}(\eta_0, n_0) + (\mathrm{sn}(\xi_1, m_1) + \mathrm{sn}(\xi_2, m_2))\mathrm{sn}(\eta_1, n_1))^2}, \tag{11}$$

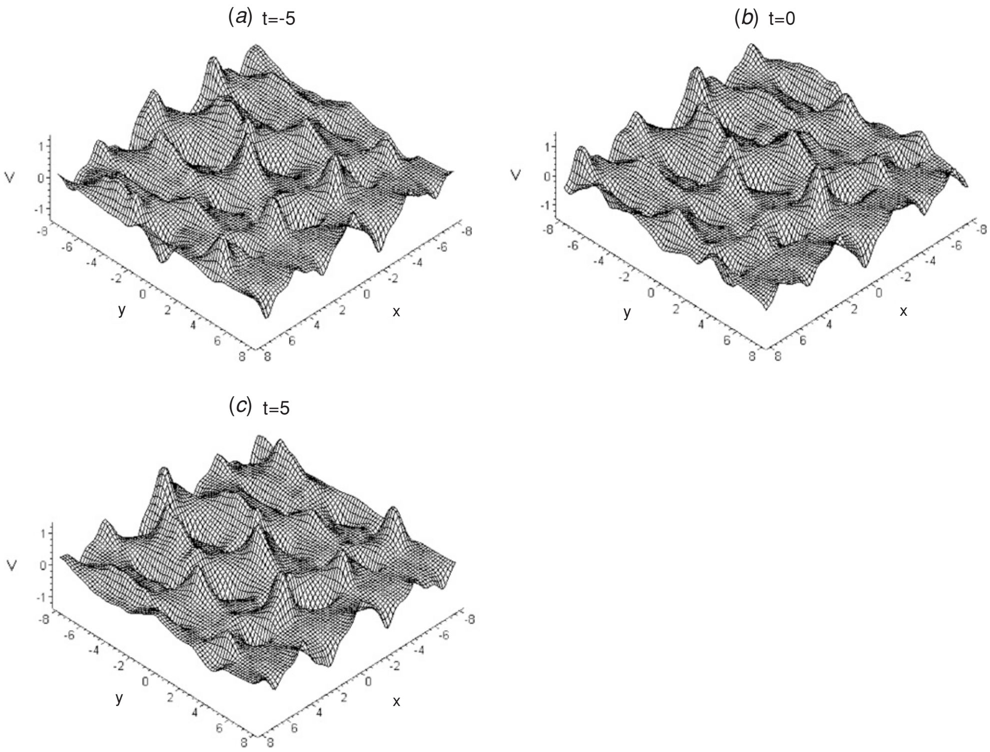

Figure 1. The evolution of a special doubly periodic structure of the Burgers for the quantity V expressed by equation (10) with the special fixed condition (11) at times (*a*) $t = -5$, (*b*) $t = 0$ and (*c*) $t = 5$, respectively.

which denotes a special type of doubly periodic wave solution in both x and y directions. Figure 1 shows the interaction property of equation (11) with different speeds. From figure 1, we can see that the interaction of doubly periodic waves is nonelastic. Moreover, p, q_0 and q_1 can also be taken as the elliptic functions of different types, and we can even take p as the form $\sum_{i=1}^{M} \mathrm{sn}(\xi_i, m_i)$, q_0 and/or q_1 the form $\sum_{j=1}^{N} \mathrm{cn}(\eta_j, n_j)$, etc. Of course, we may obtain a diversity of periodic wave solutions in terms of the Jacobi elliptic functions by selecting the arbitrary functions appropriately. It is worth noting that the Jacobi transformation, detailed description can be found in [12], implies that any solution found by one Jacobi elliptic function may be transformed into an equivalent one that can be obtained by another. Moreover, since other Jacobi elliptic functions have singularities, we consider only the periodic wave solutions in terms of sn- and cn-functions.

Remark. Due to the limit of space, we only list the sn-function expression in this paper and omit the cn-function expression and sn- and cn-functions mixed expression.

Due to the elementary properties of the elliptic functions, when $m \to 0$, the Jacobi elliptic functions degenerate to the trigonometric functions, i.e., $\mathrm{sn}\,\xi \to \sin\xi$, $\mathrm{cn}\,\xi \to \cos\xi$, $\mathrm{dn}\,\xi \to 1$, and when $m \to 1$, the Jacobi elliptic functions degenerate to the hyperbolic functions, i.e., $\mathrm{sn}\,\xi \to \tanh\xi$, $\mathrm{cn}\,\xi \to \mathrm{sech}\,\xi$, $\mathrm{dn}\,\xi \to \mathrm{sech}\,\xi$. As m_1, m_2, n_0 and n_1 approach 1, it follows from equation (11) that

$$V = \frac{2(k_1\,\mathrm{sech}^2(\xi_1) + k_2\,\mathrm{sech}^2(\xi_2))(l_0\tanh(\eta_1)\mathrm{sech}^2(\eta_0) - l_1(\alpha + \tanh(\eta_0))\mathrm{sech}^2(\eta_1))}{(\alpha + \mathrm{sn}(\eta_0, n_0) + (\mathrm{sn}(\xi_1, m_1) + \mathrm{sn}(\xi_2, m_2))\mathrm{sn}(\eta_1, n_1))^2}, \quad (12)$$

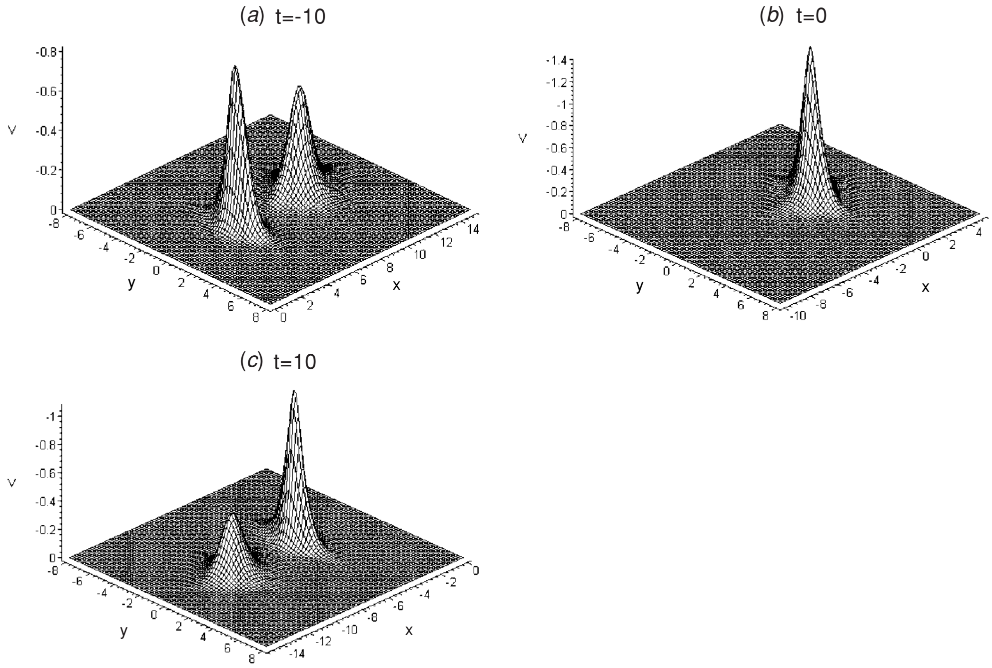

Figure 2. The evolution of two-dromion solution equation (12) with $\alpha = 5$ at times (a) $t = -10$, (b) $t = 0$ and (c) $t = 10$, respectively.

Figure 2 shows the evolution of equation (12) with $\alpha = 5$, a two-dromion solution. From the figures, one can see the interaction of two dromions with nonelastic properties. It is interesting to see that equation (12) with $\alpha = 3$ is a dromion–solitoff solution! And its evolution is shown by figure 3. We can see from figure 3 that their interaction is also nonelastic.

Along the same line of argument and performing a similar analysis, when m_1, m_2 approach 0 and n_0, n_1 approach 1, from equation (11) one has

$$V = \frac{2(k_1\cos(\xi_1) + k_2\cos(\xi_2))(l_0\tanh(\eta_1)\mathrm{sech}^2(\eta_0) - l_1(\alpha + \tanh(\eta_0))\mathrm{sech}^2(\eta_1))}{(\alpha + \tanh(\eta_0) + (\sin(\xi_1) + \sin(\xi_2))\tanh(\eta_1))^2}, \qquad (13)$$

which is periodic in the propagating direction x and exponentially decays in y. We call it x-periodic solitary wave. As m_1, m_2 approach 1 and n_0, n_1 approach 0, from equation (11) we obtain that

$$V = \frac{2(k_1\,\mathrm{sech}^2(\xi_1) + k_2\,\mathrm{sech}^2(\xi_2))(l_0\sin(\eta_1)\cos(\eta_0) - l_1(\alpha + \sin(\eta_0))\cos(\eta_1))}{(\alpha + \sin(\eta_0) + (\tanh(\xi_1) + \tanh(\xi_2))\sin(\eta_1))^2}, \qquad (14)$$

which is periodic in the propagating direction y. The graphic representation of x- and y-periodic solitary waves expressed by equations (13) and (14) is a trivial thing by MAPLE and is omitted. In other words, the study of limit cases to these doubly periodic wave solutions indicates that the Jacobi elliptic wave solutions can be viewed as the generalization of dromions, dromion–solitoff solutions and periodic solutions.

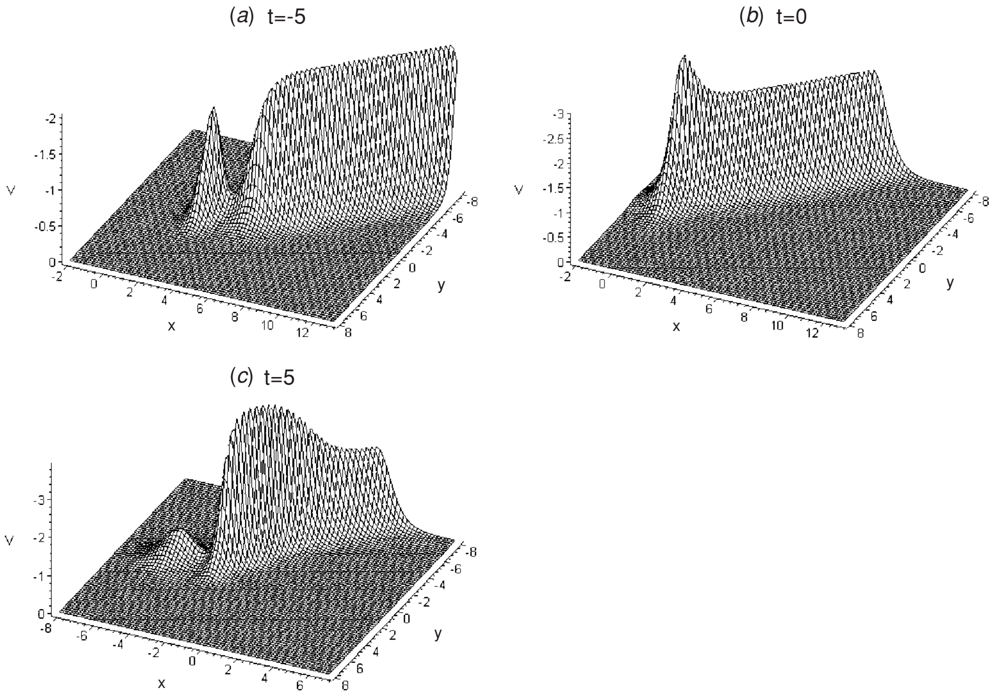

Figure 3. The evolution of dromion–solitoff solution equation (12) with $\alpha = 3$ at times (*a*) $t = -5$, (*b*) $t = 0$ and (*c*) $t = 5$, respectively.

3.2. Periodic–kink interaction waves

If we select the functions p, q_0 and q_1 such that one of them possesses localized structures while the other has a periodic structure the wave solution (10) becomes a periodic line solitary wave. For example, the selection

$$p = \mathrm{sn}(\xi_1, m_1) + \mathrm{sn}(\xi_2, m_2), \qquad \xi_1 = k_1 x + \omega_1 t, \qquad \xi_2 = k_2 x + \omega_2 t,$$
$$q_0 = 10, \qquad q_1 = \tanh(\eta_1), \qquad \eta_1 = l_1 y + c_1 t,$$

where sn is the Jacobi elliptic sine function, k_1, ω_1, k_2, ω_2, l_1, c_1, m_1 and m_2 are arbitrary constants, makes equation (10) to be a line periodic solitary wave:

$$V = \frac{-20 l_1 (k_1\,\mathrm{cn}(\xi_1, m_1)\,\mathrm{d}n(\xi_1, m_1) + k_2\,\mathrm{cn}(\xi_2, m_2)\,\mathrm{d}n(\xi_2, m_2))\mathrm{sech}^2(\eta_1)}{(10 + (\mathrm{sn}(\xi_1, m_1) + \mathrm{sn}(\xi_2, m_2))\tanh(\eta_1))^2}. \tag{15}$$

Figure 4 displays the evolution structure of equation (15) with the parameter selections as

$$k_1 = 1, \quad k_2 = 2, \quad \omega_1 = \omega_2 = 1, \quad l_1 = 1, \quad m_1 = 0.8, \quad m_2 = 0.9.$$

3.3. Periodic–peakon interaction waves

It is well known that, in addition to the continuous localized excitations in (1+1)-dimensional nonlinear systems, some types of significant weak solutions such as the peakon and compacton have attracted much attention of both mathematicians and physicists. The so-called peakon

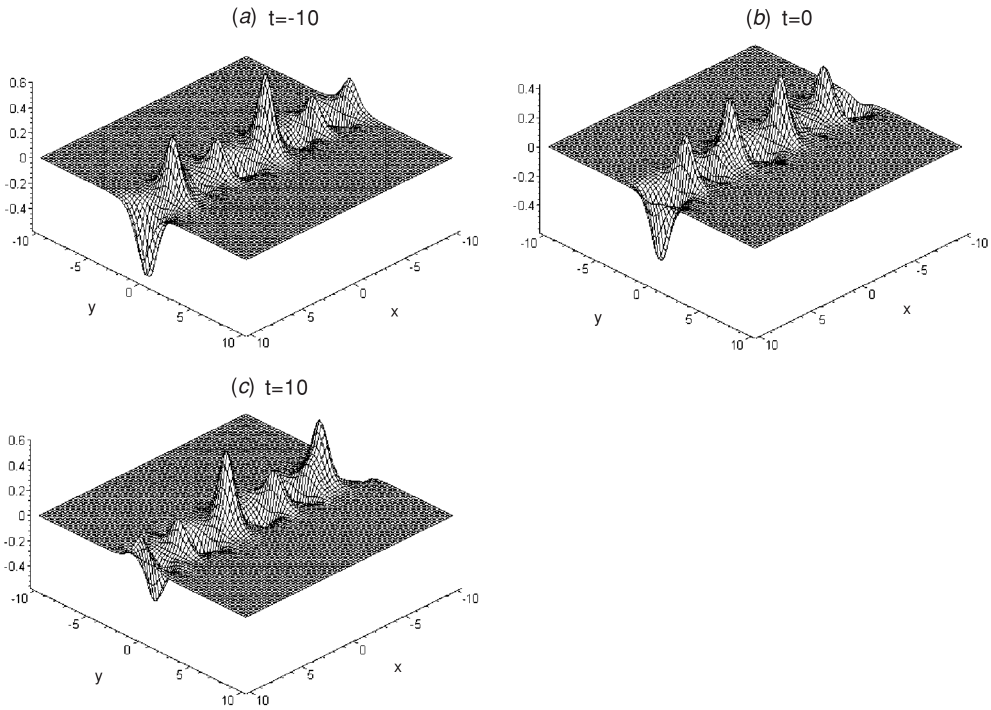

Figure 4. The evolution of the periodic–kink structure of the Burgers for the quantity expressed by equation (15) at times (*a*) $t = -10$, (*b*) $t = 0$ and (*c*) $t = 10$, respectively.

solution ($u = c\exp(-|x - ct|)$) referred to as a weak solution of the celebrated (1+1)-dimensional Camassa–Holm equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{16}$$

was firstly given by Camassa and Holm [13]. While the so-called (1+1)-dimensional compacton solutions which describe the typical (1+1)-dimensional soliton solutions with finite wavelength when the nonlinear dispersion effects are included were firstly given by Rosenau and Hyman [14]. In this subsection, we extend the peakon solution to the high-dimensional nonlinear system and give periodic–peakon wave solutions by selecting the arbitrary functions as Jacobi elliptic functions and peakon solution respectively. For instance, if we take

$$p = \begin{cases} \sum_{i=1}^{M} d_i \exp(m_i x - \beta_i t + x_{0i}), & m_i x - \beta_i t + x_{0i} \leqslant 0 \\ \sum_{i=1}^{M} (-d_i \exp(-m_i x + \beta_i t - x_{0i}) + 2d_i), & m_i x - \beta_i t + x_{0i} \rangle 0, \end{cases} \tag{17}$$

$$q_0 = 5, \qquad q_1 = \mathrm{sn}(\eta_1, n_1), \qquad \eta_1 = l_1 y + c_1 t,$$

then equation (10) is a doubly periodic–peakon wave solution.

Figure 5 exhibits the interaction structure of the periodic–peakon wave solution (17) with the parameter selections as

$$d_1 = d_2 = 1, \qquad m_1 = m_2 = 1, \qquad \beta_1 = -1, \qquad \beta_2 = 2, \qquad x_{01} = 4, \qquad x_{02} = -4, \qquad l_1 = 1, \qquad n_1 = 0.85. \tag{18}$$

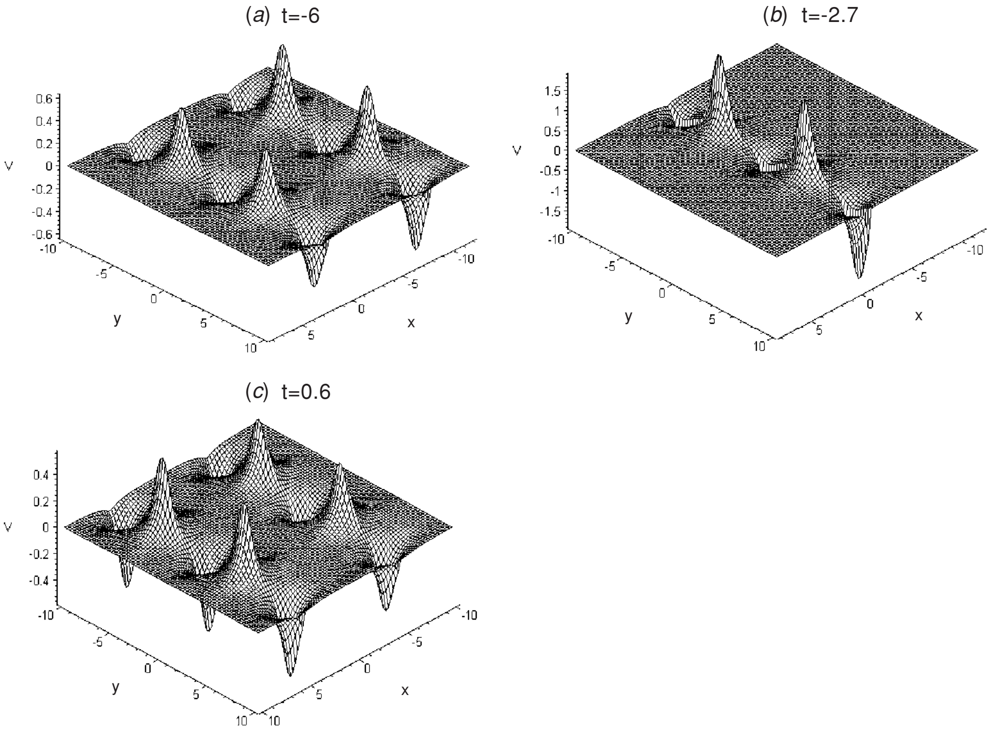

Figure 5. The evolution of a periodic–peakon interaction waves solution (10) with conditions (17) and (18) at times (*a*) $t = -6$, (*b*) $t = -2.7$ and (*c*) $t = 0.6$, respectively.

3.4. Periodic–compacton interaction waves

Similar to the above ideas, if one of the functions p, q_0 and q_1 is selected as a localized function such as compacton and the other is selected as the Jacobi elliptic function, then equation (10) becomes a periodic–compacton wave solution. Here is a special example

$$p = \sum_{i=1}^{M} \begin{cases} 0, & x + \beta_i t \leqslant x_{0i} - \dfrac{\pi}{2k_i} \\ b_i \cos^{\alpha_i+1}(k_i(x + \beta_i t - x_{0i})), & x_{0i} - \dfrac{\pi}{2k_i} < x + \beta_i t \leqslant x_{0i} + \dfrac{\pi}{2k_i} \\ 0, & x + \beta_i t > x_{0i} + \dfrac{\pi}{2k_i}, \end{cases} \tag{19}$$
$$q_0 = 5, \qquad q_1 = \mathrm{sn}(\eta_1, n_1), \qquad \eta_1 = l_1 y + c_1 t,$$

where b_i, k_i, β_i and x_{0i} are arbitrary constants, and $\{\alpha_i\}$ for all $\{i\}$ are positive integers; solution (10) becomes a periodic–compacton interaction waves.

Figure 6 displays a special periodic–compacton waves expressed by equation (10) with equation (19) and the parameter selections as

$$b_1 = -2, \qquad b_2 = -1, \qquad k_1 = k_2 = 1, \qquad \beta_1 = -1, \qquad \beta_2 = 3,$$
$$x_{01} = x_{02} = 0, \qquad \alpha_1 = \alpha_2 = 4, \qquad l_1 = 1, \qquad n_1 = 0.85. \tag{20}$$

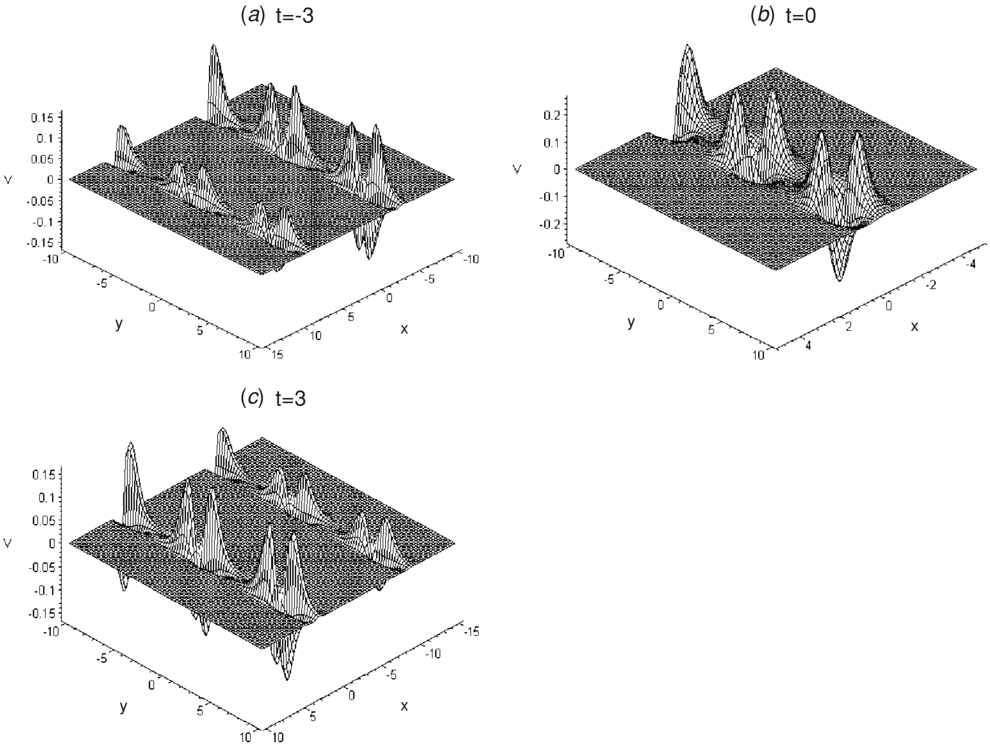

Figure 6. The evolution of periodic–compacton interaction waves as seen in the physical quantity V with conditions (19) and (20) at times (*a*) $t = -3$, (*b*) $t = 0$ and (*c*) $t = 3$, respectively.

3.5. Periodic–foldon interaction waves

As a final example along with the above ideas, we can construct periodic–foldon interaction waves for the (2+1)-dimensional Burgers system if the functions q_0 and/or q are Jacobi elliptic function and p is selected via the relations

$$p_x = \sum_{i=1}^{M} U_i(\xi + w_i t), \qquad x = \xi + \sum_{i=1}^{M} X_i(\xi + w_i t), \qquad p = \int^{\xi} p_x x_\xi \, \mathrm{d}\xi, \tag{21}$$

where U_i and X_i are localized excitations with the properties $U_i(\pm\infty) = 0$, $X_i(\pm\infty) = \text{const}$. From equation (21), one knows that ξ may be a multi-valued function in some suitable regions of x by selecting the functions X_i appropriately. Therefore, the function p_x, which is obviously an interaction solution of M localized excitations because of the property $\xi|_{x\to\infty} \to \infty$, may be a multi-valued function of x in these areas, though it is a single-valued function of ξ. Actually, most of the known multi-loop solutions are a special situation of equation (21). To study the periodic–foldon wave solutions and their interaction properties in the (2+1)-dimensional system, we take

$$\begin{aligned} &p_x = -\tfrac{4}{5}\mathrm{sech}^2(\xi) - \tfrac{1}{2}\mathrm{sech}^2(\xi - t), \qquad x = \xi - 1.5\tanh(\xi) - 1.5\tanh(\xi - t), \\ &q_0 = 5, \qquad q_1 = \mathrm{sn}(\eta_1, n_1), \qquad \eta_1 = l_1 y + c_1 t \end{aligned} \tag{22}$$

in equation (10); then the evolution plots of the periodic–foldon interaction waves are displayed in figure 7.

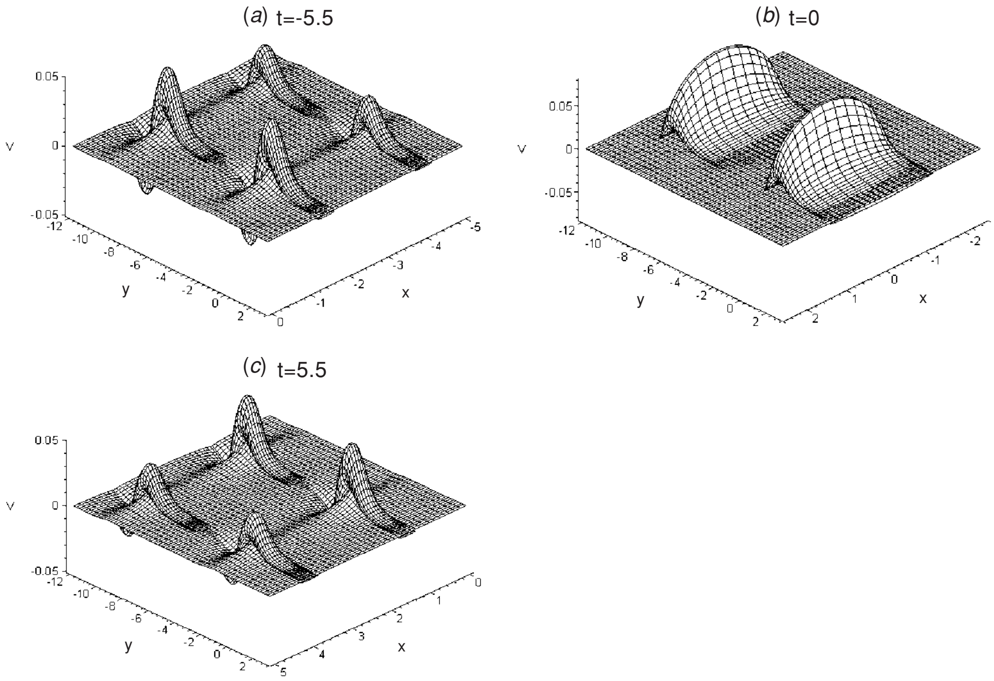

Figure 7. The evolution of periodic–foldon interaction waves as seen in the physical quantity V with conditions (22) at times (a) $t = -5.5$, (b) $t = 0$ and (c) $t = 5.5$, respectively.

4. Summary

In summary, by using the general variable separated approach, we obtain a universal formula in which arbitrary number of variable separated functions can be involved. Thanks to the existence of the arbitrary functions in the universal formula, some special types of explicit multiple wave interaction solutions including periodic–periodic waves, periodic–kink waves, periodic–peakon waves, periodic–compacton waves and periodic–foldon waves are explicitly given both analytically and graphically.

Even for two periodic interaction waves there may be many interesting features. In this paper, through the study of limit cases to these doubly periodic wave solutions, we find that the Jacobi elliptic wave solutions can be viewed as the generalization of dromions, dromion–solitoff solutions and periodic solutions. Moreover, due to the wide applications of the Burgers equation in physics, it is more interesting to find some possible applications of these exact solutions. However, lacking experiments related to the high-dimensional Burgers, we could not further say something about the real physical meanings of our exact solution. We hope that in future experimental studies some kind of exact wave solutions obtained here can be realized in some fields such as those listed in the introduction.

Acknowledgments

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